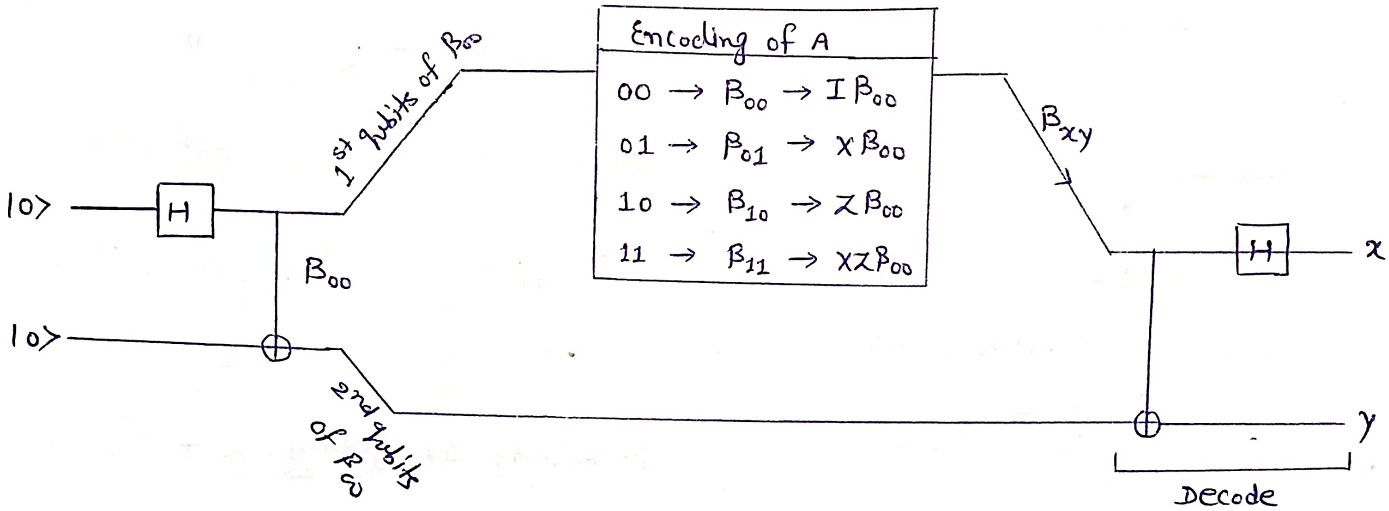
* Super-dense Coding :- "Reverse Process of Quantum Teleportation"

→ Transmit 2 bits of Classical Information by using 1 qubit of Quantum Information. (Quantum channel)



⇒ β_{00} is Encoded to β_{xy}

To send	Quantum Channel	Encoded state
00	$(I\otimes I)\beta_{00}$	$\frac{1}{\sqrt{2}}[\|00\rangle + \|11\rangle] = \beta_{00}$
01	$(X\otimes I)\beta_{00}$	$\frac{1}{\sqrt{2}}[\|01\rangle + \|10\rangle]$ β_{01}
10	$(Z\otimes I)\beta_{00}$	$\frac{1}{\sqrt{2}}[\|00\rangle - \|11\rangle]$ β_{10}
11	$(ZX\otimes I)\beta_{00}$	$\frac{1}{\sqrt{2}}[\|01\rangle - \|10\rangle]$ β_{11}

Decoding of β :

$$\beta_{00} = \frac{1}{\sqrt{2}}[|00\rangle + |11\rangle] \xrightarrow{\text{CNOT}} \frac{1}{\sqrt{2}}[|00\rangle + |10\rangle] = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|0\rangle \xrightarrow{H} |0\rangle|0\rangle$$

$$\beta_{01} = \frac{1}{\sqrt{2}}[|01\rangle + |10\rangle] \xrightarrow{\text{CNOT}} \frac{1}{\sqrt{2}}[|01\rangle + |11\rangle] = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|1\rangle \xrightarrow{H\otimes I} |0\rangle|1\rangle$$

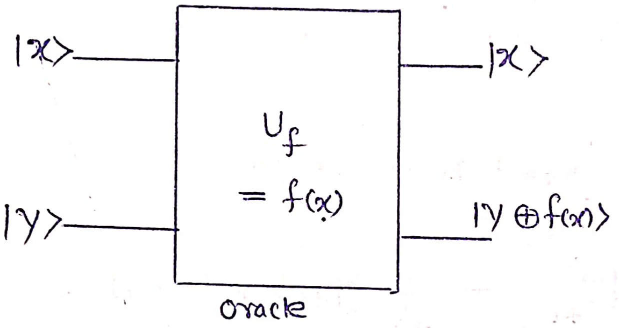
$$\beta_{10} = \frac{1}{\sqrt{2}}[|00\rangle - |11\rangle] \xrightarrow[H]{\text{CNOT}} |1\rangle|0\rangle$$

$$\beta_{11} = \frac{1}{\sqrt{2}}[|01\rangle - |10\rangle] \xrightarrow[H]{\text{CNOT}} |1\rangle|1\rangle$$

Deutsch's Algorithm — To Identify Wheather a given function is Either Balanced or Constant.

Balanced ↓
Half of the Input ⇒ takes 0
Half of the Input ⇒ takes 1

Constant ↓
Always op = 0 / 1 ∀ Input



$f : \{0,1\} \rightarrow \{0,1\}$

⇒ Examine $y \oplus f(x)$, (Let $y = |-\rangle$)

i) $f(x) = 0$

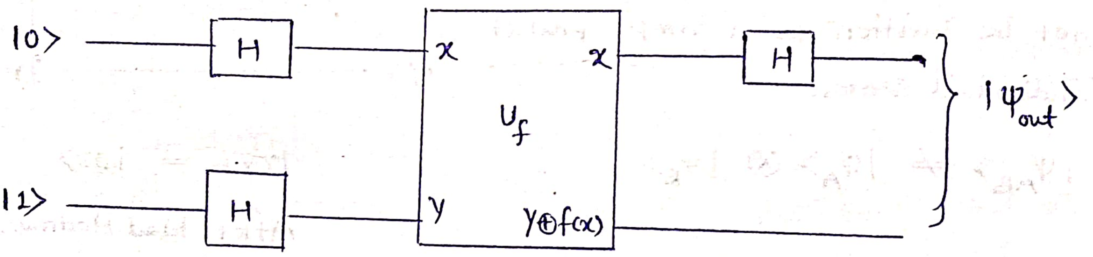
$\hookrightarrow |y \oplus 0\rangle = |- \oplus 0\rangle = |-\rangle$

ii) $f(x) = 1$

$\hookrightarrow |y \oplus 1\rangle = |- \oplus 1\rangle = -|-\rangle$

$$y \oplus f(x) = (-1)^{f(x)} |-\rangle$$

U_f Transform $|x\rangle|-\rangle \Rightarrow (-1)^{f(x)} |x\rangle |-\rangle$



$$\Rightarrow U_f \left[\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right] = \frac{1}{2}\left[(-1)^{f(0)}|0\rangle(|0\rangle - |1\rangle) + (-1)^{f(1)}|1\rangle(|0\rangle - |1\rangle)\right]$$

Case-1 : f is Constant, $f(0) = f(1)$

$$\Rightarrow \frac{1}{2}(-1)^{f(0)}\left[|0\rangle(|0\rangle - |1\rangle) + |1\rangle(|0\rangle - |1\rangle)\right]$$

$$\Rightarrow \pm\frac{1}{2}\left[|0\rangle(|0\rangle - |1\rangle) + |1\rangle(|0\rangle - |1\rangle)\right]$$

$$\Rightarrow \pm\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

$$\hookrightarrow |\psi_{out}\rangle = \pm\frac{1}{\sqrt{2}}|0\rangle(|0\rangle - |1\rangle)$$

Case-2 : f is balanced : $f(0) \neq f(1)$

$$\Rightarrow \frac{1}{2}\left[(-1)^{f(0)}|0\rangle(|0\rangle-|1\rangle) + (-1)(-1)^{f(0)}|1\rangle(|0\rangle-|1\rangle)\right]$$

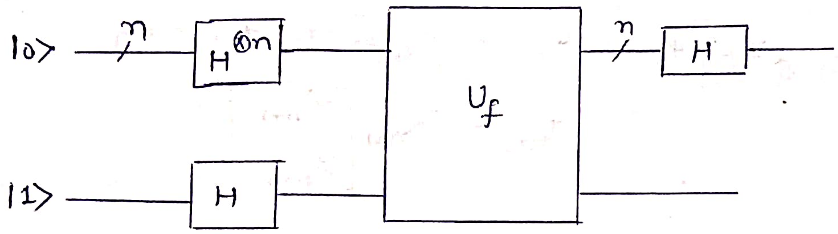
$$\Rightarrow \frac{1}{2}(-1)^{f(0)}\left[|0\rangle(|0\rangle-|1\rangle) - |1\rangle(|0\rangle-|1\rangle)\right]$$

$$\Rightarrow \pm\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)\,\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)$$

$$\hookrightarrow |\psi_{out}\rangle = \pm\frac{1}{\sqrt{2}}|1\rangle(|0\rangle-|1\rangle)$$

* Deutsch - Josza Algorithm

→ It is a generalization of deutsch's Algorithm.



$$H^{\otimes n}|0\rangle_n = \frac{1}{\sqrt{N}}\sum_{x=0}^{N-1}|x\rangle \quad , \quad N = 2^n$$

$$|\psi_{in}\rangle = H^{\otimes n}|0\rangle_n \otimes |-\rangle$$

$$= \frac{1}{\sqrt{N}}\sum_{x=0}^{N-1}|x\rangle|-\rangle \xrightarrow{U_f} \frac{1}{\sqrt{N}}\sum_{x=0}^{N-1}(-1)^{f(x)}|x\rangle|-\rangle$$

$$\Big\downarrow H^{\otimes n}\otimes I \qquad \boxed{H^{\otimes n}|x\rangle_n = \frac{1}{\sqrt{N}}\sum_{z=0}^{N-1}(-1)^{x\cdot z}|z\rangle}$$

$$\left(\frac{1}{\sqrt{N}}\right)^2\sum_{z=0}^{N-1}\left[\sum_{x=0}^{N-1}(-1)^{f(x)}\cdot(-1)^{x\cdot z}\right]|z\rangle|-\rangle$$

$$\Rightarrow \sum_{z=0}^{N-1}\left[\frac{1}{N}\sum(-1)^{f(x)}\cdot(-1)^{x\cdot z}\right]|z\rangle_n|-\rangle$$

Let Assume - $|z\rangle = |0\rangle_n$

$(-1)^{x\cdot z} = +1$

$$\Rightarrow \quad \frac{1}{N}\sum_{x=0}^{N-1}(-1)^{f(x)}$$

If $f(x)$ is Constant

$f(x) = c$, $c \in \{0, 1\}$

$$Amp = \frac{1}{N}[\pm N]$$

$$= \pm 1$$

Probability of Amplitude for $z = |0\rangle_n$ is $-$ $\underline{1}$

If $f(x)$ is balanced

$$Amp = \frac{1}{N}\sum_{x=0}^{N-1}(-1)^{f(x)}$$

Half of $f(x) = 0$ or Half of $f(x) = 1$

$$\frac{1}{2}\left[\frac{1}{N}\sum(-1)^{0}\right] + \frac{1}{2}\left[\frac{1}{N}\sum(-1)^{1}\right]$$

$$\frac{1}{2} - \frac{1}{2} = 0$$

Probability of getting $z = |0\rangle_n$ is $-$ $\underline{0}$

* Grover's Algorithm :-

→ It is a Quantum search Algorithm that finds a marked item in an unsorted list of N items in $O(\sqrt{N})$ time. (Classic search $= O(N)$)

$$f(x) = \begin{cases} 1, & x = x_{target} \\ 0, & \text{otherwise} \end{cases}$$

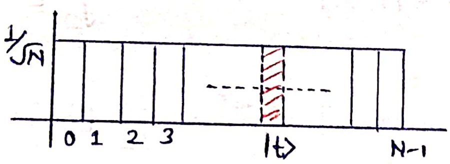
0	0	0	--- 1 ---	0

0 … N−1

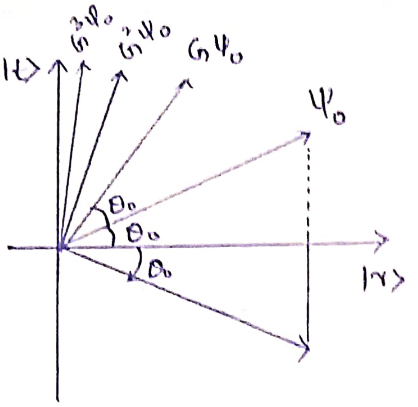
1) Create a super position state :

$|0\rangle_n$ —— $\boxed{H^{\otimes n}}$ ——

$$\psi_0 = \frac{1}{\sqrt{N}}\sum_{x=0}^{N-1}|x\rangle$$



$$\psi_0 = \alpha_0\sum_{r\neq t}|r\rangle + \beta_0|t\rangle, \quad \forall\ \alpha_0 = \beta_0 = \frac{1}{\sqrt{N}}$$



~~2.) Inversion of ψ_0 about x-axis -~~

~~↳ Inversion of Amp of Target state~~

~~$I_t = I - 2|t\rangle\langle t|$~~

~~$I_t \psi_0 = \alpha_0 \sum_{r \neq t} |r\rangle - \beta_0 |t\rangle$~~

$I_t|\psi_0\rangle = \alpha_0 \sum |r\rangle - \beta_0|t\rangle$

3) Diffusion operator -

This Invert the state with respect to any vector. 's

$$\boxed{D = 2|s\rangle\langle s| - I} = -H_n I_0 H_n$$

2.) Inversion of ψ_0 about x-axis -

↳ Inversion of Amp of Target state by oracle

$$\boxed{O = I - 2|t\rangle\langle t|}$$

$$O\psi_0 = \alpha_0 \sum_{r \neq t} |r\rangle - \beta_0 |t\rangle$$

$\rightarrow \quad G = DO$

$\Rightarrow \quad G|\psi_0\rangle \longrightarrow |\psi_f\rangle$

Amplitude of target state increasing and Amplitude of Remaining states decreasing

$$\Rightarrow \quad |\psi_f\rangle = G^r|\psi_0\rangle = (DO)^r|\psi_0\rangle$$

↳ Coincides with y-axis

$$|\psi_f\rangle = \alpha_n \sum_{r \neq t} |r\rangle + \beta_n |t\rangle$$

Normalisation Cond - $|\alpha_n|^2 (N-1) + |\beta_n|^2 = 1$

$$|\alpha_n|^2 (N-1) = \cos^2\theta_n \qquad |\beta_n|^2 = \sin^2\theta_n$$

$$\cos\theta_n = \sqrt{N-1}\,\alpha_n \qquad \sin\theta_n = \beta_n$$

$$\cos\theta_0 = \sqrt{N-1} \cdot \alpha_0 \qquad \sin\theta_0 = \beta_0$$

$$= \sqrt{\frac{N-1}{N}} \qquad = \frac{1}{\sqrt{N}} \approx \theta_0$$

After r No. of Iterations ψ_f Coincides with y axis:-

$$2r\theta_0 + \theta_0 = \pi/2$$

$$(2r+1) = \frac{\pi}{2\theta_0}$$

$$r = \frac{\pi}{4\theta_0} - \frac{1}{2}$$

$$r = \frac{\pi}{4}\sqrt{N} - \frac{1}{2} \qquad \left(\sin\theta_0 \approx \theta_0 \approx \frac{1}{\sqrt{N}}\right)$$

$$\boxed{r = O(\sqrt{N})}$$

* Matrix form of Grover's Iteration operator-

$$\psi_0 = \alpha_0 \sum_{r\neq t} |r\rangle + |t\rangle \beta_0$$

$$G^n \psi_0 = (DO)^n \psi_0 = \psi_n = \alpha_n \sum |r\rangle + |t\rangle \beta_n$$

$$\psi_{n+1} = \cancel{\alpha_{n+1}} \begin{bmatrix} \alpha_{n+1} \\ \beta_{n+1} \end{bmatrix}$$

$$O\psi_n = \alpha_n \sum_{r\neq t} |r\rangle - \beta_n |t\rangle = |V_n\rangle$$

$$D|V_n\rangle = -H_n I_0 H_n |V_n\rangle = -H_n (I - 2|0\rangle\langle 0|) H_n |V_n\rangle$$

$$= -H_n I H_n |V_n\rangle + 2 H_n |0\rangle_n \langle 0| H_n |V_n\rangle$$

$$= -|V_n\rangle + 2 \frac{1}{\sqrt{N}} \sum_x |x\rangle \frac{1}{\sqrt{N}} \sum_x \langle x|V_n\rangle$$

$$\left(|x\rangle = \sum |r\rangle + |t\rangle\right)$$

$$\psi_{n+1} = -|V_n\rangle + \frac{2}{N} \sum_x |x\rangle \left[\left(\sum \alpha_n \underbrace{\langle r|r\rangle}_{1}\right) - \beta_n \underbrace{\langle t|t\rangle}_{1}\right]$$

$$= -|V_n\rangle + \frac{2}{N} \sum_x \left[(N-1)\alpha_n - \beta_n\right] |x\rangle$$

$$= -|V_n\rangle + \cancel{\frac{2}{N}}\, 2 \sum_x \underbrace{\frac{1}{N}\left[(N-1)\alpha_n - \beta_n\right]}_{\text{Avg Amplitude} = A_n} |x\rangle$$

$$= -|V_n\rangle + 2 \sum_x A_n |x\rangle$$

$$\psi_{n+1} = -\alpha_n \sum |r\rangle + \beta_n |t\rangle + 2A_n \left(\sum_{r\neq t} |r\rangle + |t\rangle \right)$$

$$\psi_{n+1} = \underbrace{(2A_n - \alpha_n)}_{\alpha_{n+1}} \sum_{r\neq t} |r\rangle + \underbrace{(2A_n + \beta_n)}_{\beta_{n+1}} |t\rangle$$

$$\alpha_{n+1} = 2A_n - \alpha_n \qquad\qquad \beta_{n+1} = 2A_n + \beta_n$$

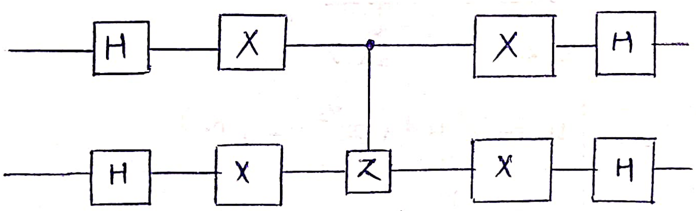
$$= \frac{2}{N}\left[\alpha_n (N-1) - \beta_n\right] - \alpha_n \qquad\qquad = \alpha_n\left(2 - \frac{2}{N}\right) + \left(1 - \frac{2}{N}\right)\beta_n$$

$$= \alpha_n\left(1 - \frac{2}{N}\right) - \frac{2}{N}\beta_n$$

$$\begin{bmatrix} \alpha_{n+1} \\ \beta_{n+1} \end{bmatrix} = \begin{bmatrix} 1-\frac{2}{N} & -\frac{2}{N} \\ 2-\frac{2}{N} & 1-\frac{2}{N} \end{bmatrix} \begin{bmatrix} \alpha_n \\ \beta_n \end{bmatrix}$$

Circuit:-

2 - qubit



* Shor's Algorithm :-

→ It is used to factor numbers into their prime components.

→ It does this in roughly $O(n^3)$ while classic computer takes exponential.

* Quantum Fourier Transformation

→ changes Computational Basis to Fourier Basis.

$$QFT\,|x\rangle = K(x,y)\,|y\rangle$$

$$\boxed{QFT\,|x\rangle = \frac{1}{\sqrt{N}} \sum_{y=0}^{N-1} e^{2\pi i \frac{xy}{N}} |y\rangle} \qquad N = 2^n$$

for 1 qubit $\Rightarrow n = 1$

$$N = 2^1 = 2$$

$$QFT\,|x\rangle = \frac{1}{\sqrt{2}} \sum_{y=0}^{1} e^{2\pi i \frac{xy}{2}} |y\rangle$$

$$|\tilde{x}\rangle = \frac{1}{\sqrt{2}} \left[|0\rangle + e^{\pi i x} |1\rangle \right]$$

$(x=0)$

$$\frac{1}{\sqrt{2}} \left[|0\rangle + |1\rangle \right] = |+\rangle$$

$(x=1)$

$$\frac{1}{\sqrt{2}} \left[|0\rangle - |1\rangle \right] = |-\rangle$$

$$\psi = \alpha|0\rangle + \beta|1\rangle$$

$$QFT\,|\psi\rangle = \alpha|\tilde{0}\rangle + \beta|\tilde{1}\rangle$$

$$= \alpha|+\rangle + \beta|-\rangle$$

$$= \left(\frac{\alpha+\beta}{\sqrt{2}}\right)|0\rangle + \left(\frac{\alpha-\beta}{\sqrt{2}}\right)|1\rangle$$

$$\because \underset{\text{decimal}}{y} = |\underbrace{y_1 y_2 y_3 \cdots\cdots y_n}_{\text{Binary}}\rangle$$

$$= y_1 2^{n-1} + y_2 * 2^{n-2} + \cdots\cdots + y_{n-1} * 2^1 + y_n * 2^0$$

$$= \sum_{K=1}^{n} y_K \cdot 2^{n-K}$$

$$|\tilde{x}\rangle = \frac{1}{\sqrt{N}} \sum_{y} e^{2\pi i x \frac{\sum_{k=1}^{n} y_k 2^{n-k}}{N}} |y_1 y_2 y_3 \cdots y_n\rangle$$

$$\because \; e^{\sum x_i} = \prod_i e^{x_i}$$

$$|\tilde{x}\rangle = \frac{1}{\sqrt{N}} \sum_{y=0}^{N-1} \prod_{k=1}^{n} e^{2\pi i x \frac{y_k}{2^k}} |y_1 y_2 \cdots y_n\rangle$$

$$= \frac{1}{\sqrt{N}} \prod_{k=1}^{n} \left[\sum_{y_1=0}^{1} \sum_{y_2=0}^{1} \cdots \sum_{y_n=0}^{1} e^{2\pi i x \frac{y_k}{2^k}} |y_k\rangle \right]$$

$$= \frac{1}{\sqrt{N}} \bigotimes_{k=1}^{n} \left[|0\rangle + e^{2\pi i x/2^k} |1\rangle \right]$$

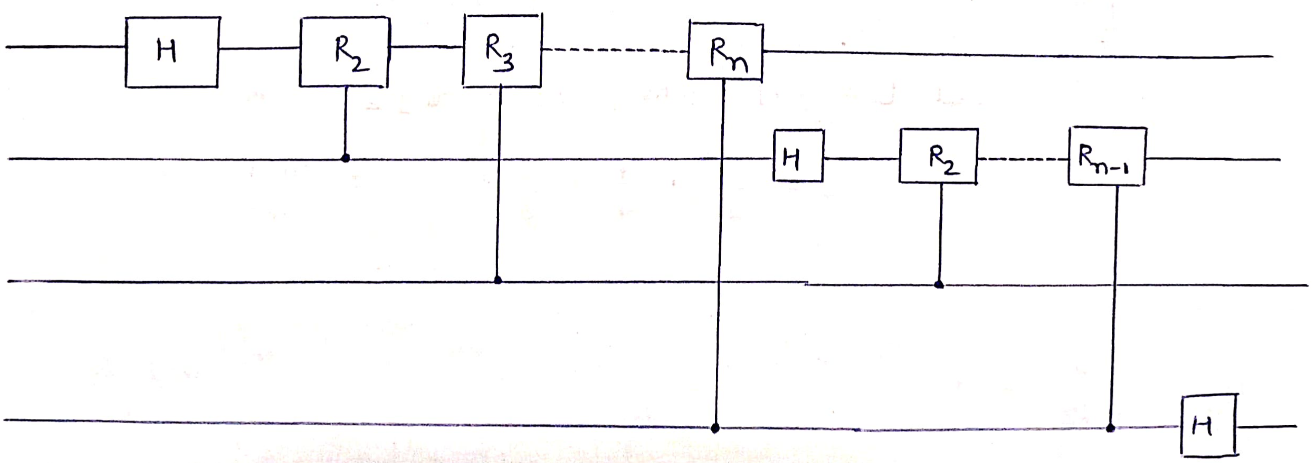
$$\because \; \frac{x}{2^n} = \frac{x_1 x_2 x_3 \cdots x_n}{2^n}$$

$$= \frac{1}{2^n} \left[x_1 * 2^{n-1} + x_2 * 2^{n-2} + \cdots\cdots + x_{n-1} 2^1 + x_n * 2^0 \right]$$

$$= (0.x_1 x_2 x_3 \cdots x_n)_2$$

$$\Rightarrow \; e^{2\pi i \frac{x}{2^n}} = e^{2\pi i (0.x_1 x_2 x_3 \cdots x_n)}$$

$$|\tilde{x}\rangle = \frac{1}{\sqrt{N}} \left[\left(|0\rangle + e^{2\pi i \frac{x}{2}} |1\rangle\right) \otimes \left(|0\rangle + e^{2\pi i \frac{x}{2^2}} |1\rangle\right) \otimes \cdots \otimes \left(|0\rangle + e^{2\pi i \frac{x}{2^n}} |1\rangle\right) \right]$$



* Bernstain - Vazirani Algorithm :-

→ Assume an unknown function $f(x) = \underline{a \cdot x}$

↳ Bitwise binary dot product

a: Hidden Variable

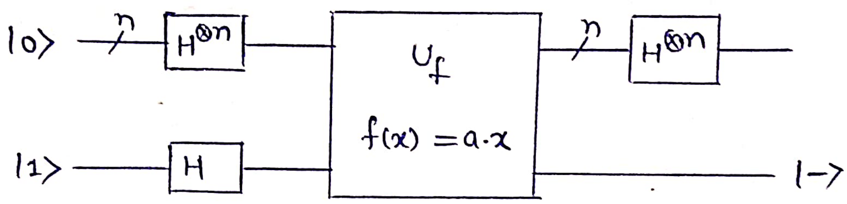
Bitwise binary dot product

↳ $x = x_1 x_2 x_3, \quad y = y_1 y_2 y_3$

$$x \cdot y = x_1 y_1 + x_2 y_2 + x_3 y_3 \pmod 2$$

$$\text{Exa} = (110) \cdot (101) = 1 + 0 + 0 = 1 \bmod 2 = 1$$

→ Aim : To find "a"



Phase Kick back :

$$|x\rangle|-\rangle \xrightarrow{U_f} (-1)^{f(x)} \cdot |x\rangle|-\rangle$$

$$= (-1)^{a \cdot x} |x\rangle|-\rangle$$

$$\Rightarrow \psi_0 = |0\rangle_n |1\rangle$$

$$\psi_1 = H^{\otimes n} |0\rangle_n |-\rangle$$

$$= \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} |x\rangle|-\rangle \xrightarrow{U_f} \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} (-1)^{a \cdot x} |x\rangle|-\rangle$$

$$\Big\downarrow H^{\otimes n} \otimes I$$

$$\frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} (-1)^{ax} \sum_{z} (-1)^{xz} |z\rangle_n |-\rangle$$

$$\Rightarrow \frac{1}{N}\sum_{z=0}^{N-1}\left[\sum_{x=0}^{N-1}(-1)^{a\cdot x}(-1)^{x\cdot z}\right]|z\rangle_n|-\rangle$$

if $|z\rangle_n = |a\rangle_n$

$$(-1)^{ax}\cdot(-1)^{ax} = (-1)^{2ax} = \underline{+1}$$

Probability of getting o/p $|z\rangle = |a\rangle_n \Rightarrow \frac{1}{N}\sum_{z=0}^{N-1}(1)$

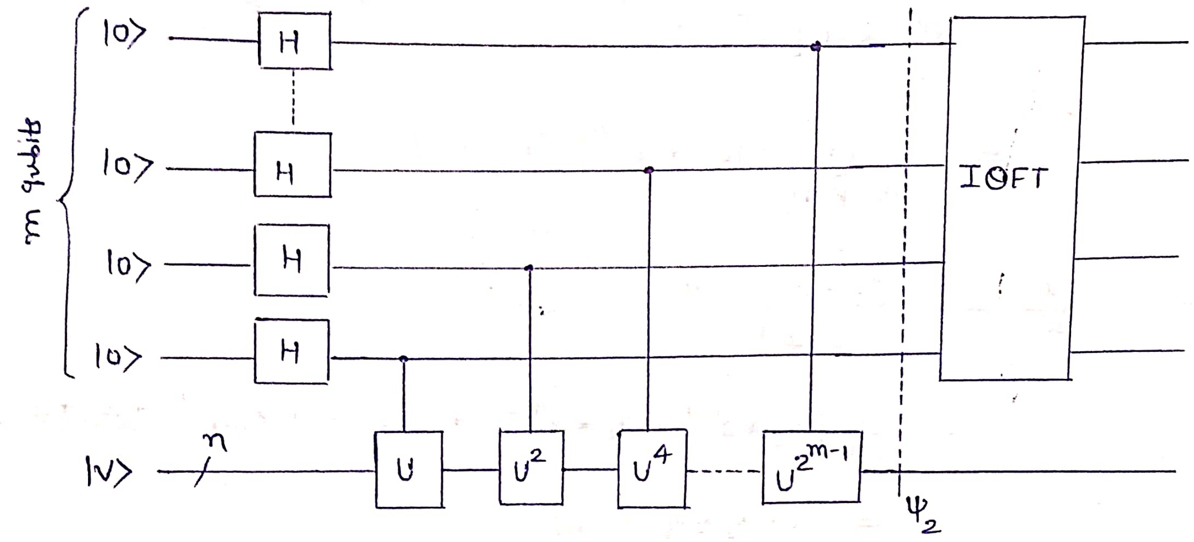
$$= \boxed{1}$$

* Phase Estimation :-

→ This is a Quantum Algorithm to estimate the phase corresponding to an eigenvalue of a given unitary operator.

Let $|v\rangle$ is a Eigenvector, with $e^{i\theta}$ Eigen value

$$U|v\rangle = e^{i\theta}|v\rangle$$



$$|\psi_0\rangle = |0\rangle^{\otimes m}|v\rangle$$

$$|\psi_1\rangle = \left[\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle+|1\rangle) \otimes \cdots\cdots \otimes \frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)\right]|v\rangle$$

$$|\psi_2\rangle = \left[\frac{1}{\sqrt{2}}(|0\rangle+e^{2^{m-1}i\theta}|1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle+e^{2^{m-2}i\theta}|1\rangle) \otimes \cdots\cdots \otimes \frac{1}{\sqrt{2}}(|0\rangle+e^{i\theta}|1\rangle)\right]|v\rangle.$$

Let $\theta = 2\pi j$, where $j = 0.j_0 j_1 j_2 \cdots\cdots j_{m-1}$ and Each $j_i \in \{0,1\}$

$$j = \frac{j_0}{2} + \frac{j_1}{4} + \frac{j_2}{8} + \cdots\cdots + \frac{j_{m-1}}{2^m}$$

$$|\psi_3\rangle = \left[\frac{1}{\sqrt{2}}(|0\rangle + e^{2\pi i\left(\frac{j_{m-1}}{2}\right)}|1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + e^{2\pi i\left(\frac{j_{m-2}}{2}+\frac{j_{m-1}}{4}\right)}|1\rangle) \otimes \cdots\cdots\right.$$

$$\left.\cdots \frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i\left(\frac{j_0}{2}+\frac{j_1}{4}+\cdots\cdots+\frac{j_{m-1}}{2^m}\right)}|1\rangle\right)\right]|v\rangle$$

$$|\psi_4\rangle = \text{IQFT}|\psi_4\rangle = \underline{\underline{|j\rangle}}$$

$$\theta = 2\pi j$$

$$\text{Eigen value} = \underline{\underline{e^{i\theta}}}$$