* Density matrix

Quantum system

1) Pure states

Super Position of Normalized vectors

Exa – $|0\rangle$, $|1\rangle$

$|00\rangle$

2) mixed States

Ensemble of two or more systems

$\{(\psi_1, P_1), (\psi_2, P_2) \dots\}$

↓

Can't represent as a state vector.

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Density matrix / operators is used for Representation of mixed states.

• density operator for pure states :-

$$\rho = |\psi\rangle\langle\psi|$$

Exa – $\psi = \frac{1}{\sqrt{2}}[|0\rangle + |1\rangle] = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\rho = |\psi\rangle\langle\psi|$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$\because \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

find $\langle\sigma_z\rangle$

$$\langle\sigma_z\rangle = \langle\psi|\sigma_z|\psi\rangle$$

By using density matrix first calculate $\rho\sigma_z$

$$\rho\sigma_z = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

$$\langle\sigma_z\rangle = Tr(\rho\sigma_z)$$

$$= \frac{1}{2}(1-1) = 0$$

$$\# \langle\hat{A}\rangle = Tr(\rho\hat{A})$$

Tr : Trace of matrix

In a Bloch sphere

$$|\psi\rangle = \begin{bmatrix} \cos\theta/2 \\ e^{i\phi}\sin\frac{\theta}{2} \end{bmatrix}$$

$$\rho = |\psi\rangle\langle\psi|$$

$$= \begin{bmatrix} \cos^2\theta/2 & e^{-i\phi}\sin\frac{\theta}{2}\cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2}\cos\frac{\theta}{2} & \sin^2\frac{\theta}{2} \end{bmatrix}$$

$\because \cos^2\frac{\theta}{2} = \frac{1}{2}(1+\cos\theta)$

$\because \sin\frac{\theta}{2}\cos\frac{\theta}{2} = \frac{1}{2}\sin\theta$ $\quad \sin^2\frac{\theta}{2} = \frac{1}{2}(1+\sin\theta)$

$$= \frac{1}{2}\begin{bmatrix} 1+\cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & 1-\cos\theta \end{bmatrix}$$

$$= \frac{1}{2}\left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \underbrace{\begin{bmatrix} \cos\theta & (\cos\phi - i\sin\phi)\sin\theta \\ (\cos\phi + i\sin\phi)\sin\theta & -\cos\theta \end{bmatrix}}_{A}\right.$$

$$\rho = \frac{1}{2}\left\{\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + A\right\}$$

$$A = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\cos\theta + \begin{bmatrix}0 & \cos\phi\sin\theta - i\sin\phi\sin\theta\\ \cos\phi\sin\theta + i\sin\phi\sin\theta & 0\end{bmatrix}$$

$$\downarrow \quad \sigma_z$$

$$\Downarrow$$

$$\begin{bmatrix}0 & \cos\phi\sin\theta\\ \cos\phi\sin\theta & 0\end{bmatrix} + \begin{bmatrix}0 & -i\sin\phi\sin\theta\\ i\sin\phi\sin\theta & 0\end{bmatrix}$$

$$A = \sigma_z\cos\theta + \sin\theta\cos\phi\,\sigma_x + \sin\theta\sin\phi\,\sigma_y$$

$$\rho = \frac{1}{2}[I + \sigma_x\sin\theta\cos\phi + \sigma_y\sin\theta\sin\phi + \sigma_z\cos\theta]$$

$$\because \hat{n} = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}$$

$$\boxed{\rho = \frac{1}{2}[I + \hat{n}\cdot\vec{\sigma}]}$$

- density operator for mixed states

Q. system $= \{(P_1,\psi_1)(P_2,\psi_2)(P_3,\psi_3)\ldots\}$

$$\boxed{\rho = \sum_i P_i |\psi_i\rangle\langle\psi_i|}$$

$$\Rightarrow \sum P_i = 1$$

→ Given An operator $\hat{A}$

$$\langle\hat{A}\rangle = Tr(\rho\hat{A})$$

Exa :- Let 50% $|0\rangle$ and 50% $|1\rangle$

$$\rho = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$$

$$= \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

→ Evolution of a density operator

$$|\psi_i\rangle \longrightarrow U|\psi_i\rangle$$

for a Closed quantum system

$$\rho = \sum_i P_i|\psi_i\rangle\langle\psi_i|$$

$$\rho' = \sum_i P_i U|\psi_i\rangle\langle\psi_i|U^\dagger$$

$$\rho' = U\rho U^\dagger$$

→ Probability of result m

If the system was in state $|\psi_i\rangle$ the probability of outcome m is:

$$P(m/i) = \langle\psi_i|M_m|\psi_i\rangle$$

Using Total probability Theorem–

$$P(m) = \sum_i P_i P(m/i)$$

$$= \sum_i P_i\langle\psi_i|M_m|\psi_i\rangle$$

$$= \sum_i P_i\, tr(M_m|\psi_i\rangle\langle\psi_i|)$$

$$= tr(M_m\rho)$$

- density operator After measurement –

→ if result m is obtained, Each pure state $|\psi_i\rangle$ Collapse to :

$$|\psi_i^m\rangle = \frac{M_m|\psi_i\rangle}{\sqrt{\langle\psi_i|M_m|\psi_i\rangle}}$$

This happens With the probability

$$P(i/m) = \frac{P(m/i)\,P_i}{P_m}$$

New density operator –

$$\rho_m = \sum_i P(i/m)\,|\psi_i^m\rangle\langle\psi_i^m|$$

$$\rho_m = \sum_i P\left(\frac{i}{m}\right)\frac{M_m|\psi_i\rangle\langle\psi_i|M_m^\dagger}{\langle\psi_i|M_m|\psi_i\rangle}$$

$$\rho_m = \frac{1}{P(m)}\sum_i \cancel{P(m/i)}\,P_i\,\frac{M_m|\psi_i\rangle\langle\psi_i|M_m^\dagger}{\cancel{P(m/i)}}$$

$$\rho_m = \frac{1}{P(m)}\sum_i P_i\,M_m|\psi_i\rangle\langle\psi_i|M_m^\dagger$$

$$\rho_m = \frac{M_m\rho M_m^\dagger}{P(m)}$$

$$\boxed{\rho_m = \frac{M_m\rho M_m^\dagger}{tr(M_m\rho)}}$$

Note :-

1) $tr(\rho^2) = 1 \Rightarrow$ Pure State

2) $tr(\rho^2) < 1 \Rightarrow$ mixed state

- Reduced density operator

→ Suppose We have a Composite System of A and B.

density operator $= \rho_{AB}$

Reduced density operator for system A is defined as –

$$\rho_A = tr_B(\rho_{AB})$$

This operation averages over all possible states of system B. and leaving the avg effect on A.

Exa : $\rho_{AB} = |a_1\rangle\langle a_2| \otimes |b_1\rangle\langle b_2|$

$$Tr_B(\rho_{AB}) = |a_1\rangle\langle a_2|\; Tr(|b_1\rangle\langle b_2|)$$